

The dual superconformal surface

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Abstract

It is shown that a superconformal surface with arbitrary codimension in flat Euclidean space has a (necessarily unique) dual superconformal surface if and only if the surface is S-Willmore, the latter a well-known necessary condition to allow a dual as shown by Ma [12]. Duality means that both surfaces envelope the same central sphere congruence and are conformal with the induced metric. Our main result is that the dual surface to a superconformal surface can easily be described in parametric form in terms of a parametrization of the latter. Moreover, it is shown that the starting surface is conformally equivalent, up to stereographic projection in the nonflat case, to a minimal surface in a space form (hence, S-Willmore) if and only if either the dual degenerates to a point (flat case) or the two surfaces are conformally equivalent (nonflat case).

A surface $f: M^2 \rightarrow \mathbb{R}^{n+2}$ in Euclidean space with codimension $n \geq 2$ is called *superconformal* if at any point the ellipse of curvature is a nondegenerate circle. Recall that the *ellipse of curvature* at $p \in M^2$ is the ellipse in the normal space $N_p M$ of f at p given by

$$\mathcal{E}(p) = \{\alpha_f(X, X) : X \in T_p M \text{ and } |X| = 1\},$$

where α_f denotes the second fundamental form of f with values in the normal bundle; see [10] and references therein for several facts on this concept whose study started almost a century ago due to the work of Moore and Wilson [14], [15].

Superconformality is invariant under conformal transformations since the property of $\mathcal{E}(p)$ being a circle is invariant under conformal changes of the metric of the ambient space. Hence, the results in this paper belong to the realm of conformal (Möbius) geometry of surfaces and can also be stated in terms of surfaces in a space form.

It was shown by Rouxel [16] that superconformal surfaces in codimension two always arise in pairs $f, \tilde{f}: M^2 \rightarrow \mathbb{R}^4$ of dual surfaces that induce conformal metrics on M^2 and envelop a common central sphere congruence. Recall that the *central sphere congruence* (or *mean curvature sphere congruence*) of an Euclidean surface with any codimension is the family of two-dimensional spheres that are tangent to the surface and have the same mean curvature vector as the surface at the point of tangency. The concept of central sphere congruence (called the *conformal Gauss map* in a different context by Bryant

[5]) is central in conformal geometry and was extensively studied since the turn of the last century, fundamentally due to the work of Thomsen [17] and Blaschke [1]; see [11] for a detailed discussion of this subject.

Rouxel also discovered that the surface of centers of the central sphere congruence is a minimal surface of \mathbb{R}^4 . If f is free of minimal points, the *surface of centers* is the locus of centers of the spheres in the congruence, thus parametrically described by the map $g: M^2 \rightarrow \mathbb{R}^{n+2}$ given by

$$g = f + \frac{1}{|H|^2} H$$

where H denotes the mean curvature vector field of f .

In this paper, we consider superconformal surfaces in Euclidean space in arbitrary codimension. To no surprise, the case of codimension two is rather special and this has much to do with the minimality of the surface of centers. In fact, this property and the classical Weierstrass representation of minimal surfaces allowed Dajczer and Tojeiro [7] to provide a complete local parametric representation of all superconformal surfaces in \mathbb{R}^4 . Moreover, they showed that the dual to a superconformal surface in codimension two reduces to a point if and only if the surface is conformally equivalent, i.e., congruent by a conformal diffeomorphism of \mathbb{R}^4 , to a holomorphic curve in \mathbb{C}^2 .

By a *dual* to a surface $f: M^2 \rightarrow \mathbb{R}^{n+2}$ we mean an immersion $\tilde{f}: M^2 \rightarrow \mathbb{R}^{n+2}$ that induces a conformal metric and possess a common central sphere congruence, that is, at each point of M^2 the sphere in the their centrals sphere congruences is the same. In fact, for convenience we allow the dual to reduce to a single point.

For a locally conformally substantial superconformal surface in codimension higher than two that carries a dual superconformal surface, it turns out that the surface of centers is *never* minimal. A surface being *locally conformally substantial* means that the image under f of any open subset of M^2 is not contained in a proper affine subspace or a sphere in the ambient space \mathbb{R}^{n+2} . This and the fact that in higher codimension superconformality is not longer such a strong assumption, make unlikely the goal to obtain, a complete parametric classification as in [7]. Nevertheless, it seems natural to expect for some class of superconformal surfaces the existence of a dual surface similar to the case considered by Rouxel. In fact, this turns out to be the case for the superconformal surfaces that are S-Willmore.

The concept of S-Willmore was introduced by Ejiri [9] as a special class of Willmore surfaces. Ma [12] showed that being S-Willmore is the condition for a surface to have a dual that, in fact, is unique. For a complex coordinate $z = y_1 + iy_2$ associate to local isothermal coordinates superconformality means that the complex line bundle spanned by $\alpha_f(\partial_z, \partial_{\bar{z}})$ is isotropic and S-Willmore that it is holomorphic with respect to the normal connection.

It is well-known [9] that minimal surfaces in space forms are the basic examples of S-Willmore surfaces. Hence, the “trivial” examples of superconformal S-Willmore surfaces in Euclidean space are the ones conformally equivalent to minimal superconformal

surfaces in Euclidean space and the images under stereographic projection of the same class of surfaces in the sphere or hyperbolic space. Euclidean minimal superconformal surfaces are called *1-isotropic* and admit a Weierstrass type representation given in [4] based on results in [3]. In the spherical case, this class of surfaces has been studied in different contexts, see [2], [13] and [18].

There are plenty of “non-trivial” examples of superconformal S-Willmore surfaces in Euclidean space. For instance, the image under stereographic projection of any super Willmore surface in an even dimensional sphere is a superconformal S-Willmore surface. The class of super Willmore surfaces was introduced and classified by Ejiri [9] in terms of isotropic holomorphic curves in complex projective spaces.

Note that in conformal geometry we may assume, at least locally, that the mean curvature of a surface never vanishes by composing with a conformal diffeomorphism.

Theorem 1. *Let $f: M^2 \rightarrow \mathbb{R}^{n+2}$, $n \geq 3$, be a regular locally conformally substantial superconformal surface. Then f has a dual superconformal surface if and only if it is S-Willmore. Moreover, the dual surface can be parametrized as*

$$\tilde{f} = f + \frac{2}{|H|^2}(H)^\Lambda,$$

where Λ is the normal subbundle of rank $n - 2$ of the surface of centers perpendicular to the plane subbundle of the first normal bundle N_1^f of f orthogonal to the mean curvature vector and $(H)^\Lambda$ denotes taking the Λ -component. Furthermore, up to conformal equivalence, we have the following cases:

- (i) *The dual reduces to a single point if and only if f is a minimal surface.*
- (ii) *The dual is obtained by composing f with an inversion and a reflection with respect to its center if and only if f is the image under stereographic projection of a minimal surface in the sphere \mathbb{S}^{n+2} .*
- (iii) *The dual is obtained by composing f with an inversion if and only if f is the image under stereographic projection of a minimal surface in the hyperbolic space \mathbb{H}^{n+2} .*

The necessity of the surface being S-Willmore in the theorem is due to Ma [12] as already mentioned. A submanifold being *regular* (or nicely curved) means that the first normal spaces, i.e., the normal subspaces spanned by the second fundamental form, have constant dimension and thus form a subbundle of the normal bundle. Notice that any isometric immersion is regular along the connected components of an open dense subset of the manifold, hence in local submanifold theory, as is the case of this paper, regularity is just a minor technical assumption. Finally, we mention that part (i) is known (see Remark on p. 339 of [9]) but we were not able to find a proof.

Any superconformal surface in codimension two is S-Willmore, thus there is no need of such requirement in that case. The codimension three case is still quite special as shown by the following result.

Theorem 2. *Any superconformal Willmore surface $f: M^2 \rightarrow \mathbb{R}^5$ is S-Willmore.*

The paper concludes with a proof of the main result in [7] by means of the approach we developed here.

1 Preliminaries

In this section, we first recall some basic properties of the ellipse of curvature of a surface and then briefly discuss the notions of superconformal and S-Willmore surface.

Let $f: M^2 \rightarrow \mathbb{R}^{n+2}$, $n \geq 2$, stand for an isometric immersion of a two-dimensional Riemannian manifold into Euclidean space. Denote by $\alpha_f: TM \times TM \rightarrow N_f M$ its second fundamental form taking values in the normal bundle.

Given an orthonormal basis $\{X_1, X_2\}$ of the tangent space $T_p M$ at $p \in M^2$, denote $\alpha_{ij} = \alpha_f(X_i, X_j)$, $1 \leq i, j \leq 2$. Then, for any unit vector $v = \cos \theta X_1 + \sin \theta X_2$ we have

$$\alpha_f(v, v) = H + \cos 2\theta \xi_1 + \sin 2\theta \xi_2, \quad (1)$$

where $\xi_1 = \frac{1}{2}(\alpha_{11} - \alpha_{22})$, $\xi_2 = \alpha_{12}$ and $H = \frac{1}{2}(\alpha_{11} + \alpha_{22})$ is the mean curvature vector of f at p . Thus, when v goes once around the unit tangent circle, the vector $\alpha_f(v, v)$ goes twice around the ellipse of curvature $\mathcal{E}(p)$ of f at p centered at H . Clearly $\mathcal{E}(p)$ degenerates into a line segment or a point if and only if ξ_1 and ξ_2 are linearly dependent, that is, at points where the normal curvature tensor R^\perp vanishes. It follows from (1) that $\mathcal{E}(p)$ is a circle if and only if for some (and hence any) orthonormal basis of $T_p M$ it holds that

$$\langle \alpha_{12}, \alpha_{11} - \alpha_{22} \rangle = 0 \quad \text{and} \quad |\alpha_{11} - \alpha_{22}| = 2|\alpha_{12}|.$$

The complexified tangent bundle $TM \otimes \mathbb{C}$ is decomposed into the eigenspaces of the complex structure J , denoted by $T' M$ and $T'' M$, corresponding to the eigenvalues i and $-i$. The complex structure of M^2 is determined by the orientation and the induced metric. The second fundamental form can be complex linearly extended to $TM \otimes \mathbb{C}$ with values in the complexified vector bundle $N_f M \otimes \mathbb{C}$ and then decomposed into its (p, q) -components, $p + q = 2$, which are tensor products of p many 1-forms vanishing on $T'' M$ and q many 1-forms vanishing on $T' M$.

Taking local isothermal coordinates $\{y_1, y_2\}$ and $z = y_1 + iy_2$, we have that the surface f is superconformal if and only if the $(2, 0)$ -part of the second fundamental form is isotropic, or equivalently, if the complex line bundle $\alpha_f(\partial_z, \partial_z)$ is isotropic. A surface

$f: M^2 \rightarrow \mathbb{R}^{n+2}$ is called *S-Willmore* [9], [12] when the complex line bundle $\alpha_f(\partial_z, \partial_z)$ is parallel in the normal bundle, that is, if

$$\nabla_{\partial_z}^\perp \alpha_f(\partial_z, \partial_z) \text{ is parallel to } \alpha_f(\partial_z, \partial_z).$$

It is well-known that any S-Willmore surface is always Willmore [9] but the converse is not true (cf. [8]) unless the substantial codimension is $n = 2$. A surface being Willmore or S-Willmore is invariant under conformal diffeomorphisms of Euclidean space. Recall that a surface $f: M^2 \rightarrow \mathbb{R}^{n+2}$ is called *Willmore* [9] if its mean curvature vector field H satisfies the Willmore surface equation obtained as the Euler-Lagrange equation of the Willmore functional, namely, if

$$\Delta^\perp H - 2|H|^2 H + \sum_{i,j=1}^2 \langle H, \alpha_{ij} \rangle \alpha_{ij} = 0 \quad (2)$$

where Δ^\perp is the Laplacian in $N_f M$ and X_1, X_2 is an orthonormal frame.

Using the Codazzi equation, it follows that

$$\nabla_{\partial_z}^\perp H = \frac{2}{\rho^2} \nabla_{\partial_z}^\perp \alpha_f(\partial_z, \partial_z),$$

where $ds^2 = \rho^2 |dz|^2$ is the induced metric. Thus, the surface is S-Willmore if and only if $\nabla_{\partial_z}^\perp H$ is parallel to $\alpha_f(\partial_z, \partial_z)$ or, equivalently, if

$$\nabla_V^\perp H \text{ is parallel to } \alpha_f(V, V) \quad (3)$$

for any $V \in T'M$.

2 The proofs

We proceed with the proofs of the results stated in the introduction. We caution that several arguments contain simple but long computations denominated straightforward that may be only sketched.

In the sequel we denote by $f: M^2 \rightarrow \mathbb{R}^{n+2}$, $n \geq 2$, a regular locally substantial superconformal surface. The latter assumption is that the image under f of any open subset of M^2 is not contained in a proper affine subspace of the ambient space. Recall that *regular* means that the first normal spaces have constant dimension and thus form a subbundle of the normal bundle. The *first normal space* N_1^f of f at $p \in M^2$ is the normal subspace spanned by the second fundamental form, i.e.,

$$N_1^f(p) = \text{span}\{\alpha_f(X, Y) : X, Y \in T_p M\}.$$

Under the above assumptions, it is easy to see that second fundamental form of the surface has the shape

$$A_{\xi_1} = \begin{pmatrix} \lambda_1 + \mu & 0 \\ 0 & \lambda_1 - \mu \end{pmatrix}, \quad A_{\xi_2} = \begin{pmatrix} \lambda_2 & \mu \\ \mu & \lambda_2 \end{pmatrix} \quad \text{and} \quad A_\delta = \lambda I$$

with respect to orthonormal frames $\{X_1, X_2\}$ of the tangent bundle and $\{\xi_1, \xi_2, \delta\}$ of the first normal subbundle N_1^f . Thus the mean curvature vector field of f is

$$H = \lambda_1 \xi_1 + \lambda_2 \xi_2 + \lambda \delta.$$

Notice that we cannot have that $\mu = 0$ on an open subset of M^2 since, otherwise, f would be totally umbilical along that set and this contradicts being substantial. In particular, the special case $\dim N_1^f = 2$ (in particular, if $n = 2$) can only occur if $\lambda = 0$.

From the Codazzi equations for ξ_1, ξ_2 and δ we obtain, respectively, that

$$\begin{aligned} X_1(\lambda_1) - X_1(\mu) &= -2\mu\Gamma_2 - \mu\gamma_2 + \lambda_2\gamma_1 - \lambda\psi_{11}, & X_2(\lambda_1) + X_2(\mu) &= 2\mu\Gamma_1 - \mu\gamma_1 + \lambda_2\gamma_2 - \lambda\psi_{21}, \\ X_1(\mu) - X_2(\lambda_2) &= 2\mu\Gamma_2 + \gamma_2(\lambda_1 + \mu) + \lambda\psi_{22}, & X_2(\mu) - X_1(\lambda_2) &= 2\mu\Gamma_1 + \gamma_1(\lambda_1 - \mu) + \lambda\psi_{12}, \end{aligned}$$

and

$$X_1(\lambda) = \psi_{11}(\lambda_1 - \mu) + \lambda_2\psi_{12} - \mu\psi_{22}, \quad X_2(\lambda) = \psi_{21}(\lambda_1 + \mu) + \lambda_2\psi_{22} - \mu\psi_{12}, \quad (4)$$

where we used the following notations:

$$\Gamma_i = \langle \nabla_{X_i} X_i, X_j \rangle, \quad i \neq j, \quad \gamma_i = \langle \nabla_{X_i}^\perp \xi_1, \xi_2 \rangle \quad \text{and} \quad \psi_{ij} = \langle \nabla_{X_i}^\perp \delta, \xi_j \rangle, \quad i, j = 1, 2.$$

The first four equations yield

$$\begin{aligned} X_1(\lambda_1) - X_2(\lambda_2) &= \lambda_1\gamma_2 + \lambda_2\gamma_1 + \lambda(\psi_{22} - \psi_{11}), \\ X_2(\lambda_1) + X_1(\lambda_2) &= -\lambda_1\gamma_1 + \lambda_2\gamma_2 - \lambda(\psi_{21} + \psi_{12}). \end{aligned}$$

Setting

$$X_1(\lambda_1) = \lambda_2\gamma_1 - \lambda\psi_{11} - \mu a_1, \quad X_1(\lambda_2) = -\lambda_1\gamma_1 - \lambda\psi_{12} - \mu a_2, \quad (5)$$

for some smooth functions a_1, a_2 , we obtain that

$$X_2(\lambda_1) = \lambda_2\gamma_2 - \lambda\psi_{21} + \mu a_2, \quad X_2(\lambda_2) = -\lambda_1\gamma_2 - \lambda\psi_{22} - \mu a_1, \quad (6)$$

$$X_1(\mu) = \mu(2\Gamma_2 + \gamma_2 - a_1), \quad X_2(\mu) = \mu(2\Gamma_1 - \gamma_1 - a_2). \quad (7)$$

The Codazzi equation for any $\eta \in (N_1^f)^\perp$ is equivalent to

$$\langle \nabla_{X_1}^\perp \eta, H \rangle = \mu(\langle \nabla_{X_1}^\perp \eta, \xi_1 \rangle + \langle \nabla_{X_2}^\perp \eta, \xi_2 \rangle), \quad \langle \nabla_{X_2}^\perp \eta, H \rangle = \mu(\langle \nabla_{X_1}^\perp \eta, \xi_2 \rangle - \langle \nabla_{X_2}^\perp \eta, \xi_1 \rangle).$$

Let $\{\eta_\alpha\}_{1 \leq \alpha \leq n-3}$ denote an orthonormal frame of $(N_1^f)^\perp$ and set

$$\psi_{ij}^\alpha = \langle \nabla_{X_i}^\perp \eta_\alpha, \xi_j \rangle, \quad 1 \leq i, j \leq 2.$$

If $\dim N_1^f = 2$ then $1 \leq \alpha \leq n-2$. In the sequel, we work the case $\dim N_1^f = 3$, but most of the computations hold if $\dim N_1^f = 2$. For simplicity, we denote

$$\psi_1 = \psi_{11} + \psi_{22}, \quad \psi_2 = \psi_{21} - \psi_{12} \quad \text{and} \quad \psi_1^\alpha = \psi_{11}^\alpha + \psi_{22}^\alpha, \quad \psi_2^\alpha = \psi_{21}^\alpha - \psi_{12}^\alpha.$$

It follows using (4) and (5) to (7) that

$$\nabla_{X_1}^\perp H = -\mu(a_1\xi_1 + a_2\xi_2 + \psi_1\delta + \Sigma_\alpha\psi_1^\alpha\eta_\alpha), \quad \nabla_{X_2}^\perp H = \mu(a_2\xi_1 - a_1\xi_2 + \psi_2\delta + \Sigma_\alpha\psi_2^\alpha\eta_\alpha). \quad (8)$$

The locus of the centers of the central sphere congruence given by the map $g: M^2 \rightarrow \mathbb{R}^{n+2}$ defined as

$$g = f + r^2 H, \quad \text{where } r = 1/|H|,$$

satisfies

$$g_* Z = f_*(I - r^2 A_H)Z + r^2 \nabla_Z^\perp H + Z(r^2)H \quad (9)$$

where

$$A_H = \begin{pmatrix} |H|^2 + \lambda_1\mu & \lambda_2\mu \\ \lambda_2\mu & |H|^2 - \lambda_1\mu \end{pmatrix}.$$

Using that

$$A_H^2 = 2|H|^2 A_H - (|H|^4 - \mu^2\theta)I$$

where $\theta = \lambda_1^2 + \lambda_2^2 = |H|^2 - \lambda^2$, it follows that

$$\langle g_* Z, g_* Y \rangle = r^4 \mu^2 \theta \langle Z, Y \rangle + r^4 \langle \nabla_Z^\perp H, \nabla_Y^\perp H \rangle.$$

$$\text{Thus } f \text{ and } g \text{ are conformal} \iff |\nabla_{X_1}^\perp H| = |\nabla_{X_2}^\perp H| \text{ and } \langle \nabla_{X_1}^\perp H, \nabla_{X_2}^\perp H \rangle = 0. \quad (10)$$

Proposition 3. *The following facts are equivalent:*

- (i) *The immersion f is S-Willmore.*
- (ii) *The immersions f and g are conformal and $\nabla^\perp H \subset N_1^f$.*
- (iii) *$\nabla^\perp H \subset \text{Im}(\alpha_f - \langle \cdot, \cdot \rangle H)$.*
- (iv) *$\psi_1 = \psi_2 = 0$ and $\psi_1^\alpha = \psi_2^\alpha = 0$, $1 \leq \alpha \leq n-3$.*

Proof: On one hand,

$$\alpha_f(X_1 - iX_2, X_1 - iX_2) = 2\mu(\xi_1 - i\xi_2).$$

On the other hand, we have from (8) that

$$\frac{1}{\mu} \nabla_{X_1 - iX_2}^\perp H = -(a_1 + ia_2)(\xi_1 - i\xi_2) - (\psi_1 + i\psi_2)\delta - \Sigma_\alpha(\psi_1^\alpha + i\psi_2^\alpha)\eta_\alpha,$$

and it follows from (3) that (i) and (iv) are equivalent.

From (8) we see that the right hand side of (10) is equivalent to

$$\psi_1^2 + \Sigma_\alpha(\psi_1^\alpha)^2 = \psi_2^2 + \Sigma_\alpha(\psi_2^\alpha)^2 \text{ and } \psi_1\psi_2 + \Sigma_\alpha\psi_1^\alpha\psi_2^\alpha = 0, \quad (11)$$

and the remaining of the argument follows easily from (8) to (11). ■

Corollary 4. *If f is S -Willmore then $(N_1^f)^\perp \subset N_g M$.*

Proof: We have from (9) that $\langle g_* Z, \eta_\alpha \rangle = 0$, $1 \leq \alpha \leq n-3$, for any $Z \in TM$. ■

We now prove the second result stated in the introduction.

Proof of Theorem 2: The Ricci equation

$$\langle R^\perp(X_1, X_2)H, \xi_j \rangle = \langle [A_H, A_{\xi_j}]X_1, X_2 \rangle, \quad j = 1, 2,$$

together with (7) and (8) yield for $j = 1$ that

$$X_1(a_2) + X_2(a_1) - 2a_1a_2 + a_1\Gamma_1 + a_2\Gamma_2 + \psi_{11}\psi_2 + \psi_{21}\psi_1 = 2\mu\lambda_2 \quad (12)$$

and for $j = 2$ that

$$-X_1(a_1) + X_2(a_2) + a_1^2 - a_2^2 - a_1\Gamma_2 + a_2\Gamma_1 + \psi_{12}\psi_2 + \psi_{22}\psi_1 = -2\mu\lambda_1. \quad (13)$$

On the other hand,

$$\begin{aligned} \langle \Delta^\perp H, \xi_j \rangle &= X_1 \langle \nabla_{X_1}^\perp H, \xi_j \rangle + X_2 \langle \nabla_{X_2}^\perp H, \xi_j \rangle - \langle \nabla_{X_1}^\perp H, \nabla_{X_1}^\perp \xi_j \rangle - \langle \nabla_{X_2}^\perp H, \nabla_{X_2}^\perp \xi_j \rangle \\ &\quad - \Gamma_1 \langle \nabla_{X_2}^\perp H, \xi_j \rangle - \Gamma_2 \langle \nabla_{X_1}^\perp H, \xi_j \rangle. \end{aligned}$$

Using (7), (8), (12) and (13) we easily obtain

$$\frac{1}{\mu} \langle \Delta^\perp H, \xi_1 \rangle = -2\lambda_1\mu - \psi_1^2 + \psi_2^2 \quad \text{and} \quad \frac{1}{\mu} \langle \Delta^\perp H, \xi_2 \rangle = -2\lambda_2\mu + 2\psi_1\psi_2.$$

Also,

$$\sum_{i,j=1}^2 \langle \alpha_f(X_i, X_j), H \rangle \alpha_f(X_i, X_j) = 2|H|^2 H + 2\mu^2(\lambda_1\xi_1 + \lambda_2\xi_2).$$

Now, we have from (2) that f is Willmore if and only if $\psi_1 = 0 = \psi_2$, and the result follows from Proposition 3. ■

Proposition 5. *Let $f: M^2 \rightarrow \mathbb{R}^{n+2}$ be a substantial superconformal S -Willmore surface with $\dim N_1^f = 2$. If $n \geq 3$ then f is minimal.*

Proof: The same proof given in Proposition 3 that parts (i) and (iv) are equivalent still holds if $\dim N_1^f = 2$. Thus $\psi_1^\alpha = 0 = \psi_2^\alpha$ for $1 \leq \alpha \leq n-2$. On the other hand, the Codazzi equation for η_α is

$$\psi_{11}^\alpha A_{\xi_1} X_2 + \psi_{12}^\alpha A_{\xi_2} X_2 = \psi_{21}^\alpha A_{\xi_1} X_1 + \psi_{22}^\alpha A_{\xi_2} X_1.$$

We obtain that

$$\lambda_2 \psi_{11}^\alpha - \lambda_1 \psi_{12}^\alpha = 0 \quad \text{and} \quad \lambda_1 \psi_{11}^\alpha + \lambda_2 \psi_{12}^\alpha = 0.$$

But $\theta \neq 0$ would give $\psi_{ij}^\alpha = 0$, which is not possible. Thus f is minimal. ■

Proposition 6. *Let $f: M^2 \rightarrow \mathbb{R}^{n+2}$ be a superconformal S-Willmore surface with $\theta = 0$ and $\dim N_1^f = 3$. Then f is minimal inside a sphere in \mathbb{R}^{n+2} .*

Proof: From (5) and (6) we obtain $\psi_{ij} = 0 = a_1 = a_2$. Then (4) implies that $|H|$ is constant. Since $H \neq 0$, then (8) and Proposition 3 show that the umbilical direction H is parallel in the normal connection. ■

Remark 7. It follows from (9) that the locus of the centers of the central sphere congruence of a non-minimal surface is a point if and only if the surface is minimal in a sphere.

In the sequel, we also assume that f is S-Willmore with $\theta \neq 0 \neq \lambda$ everywhere. Set

$$h_1 = r^2(\lambda_2\xi_1 - \lambda_1\xi_2), \quad h_2 = r^2H - \frac{1}{\lambda}\delta \quad \text{and} \quad h_j = \eta_{j-2}, \quad 3 \leq j \leq n-1.$$

Thus N_1^f is spanned by orthogonal vectors

$$N_1^f = \text{span}\{h_1, h_2, H\} \quad \text{where} \quad |h_1|^2 = r^4\theta \quad \text{and} \quad |h_2|^2 = \frac{r^2\theta}{\lambda^2}.$$

Lemma 8. *The following equations hold:*

$$h_{1*} = g_* \circ J + \omega_{11}h_1 + \omega_{12}h_2 + \Sigma_\alpha \omega_{1\alpha}h_\alpha, \quad (14)$$

$$h_{2*} = g_* + \omega_{21}h_1 + \omega_{22}h_2 + \Sigma_\alpha \omega_{2\alpha}h_\alpha, \quad (15)$$

$$h_{\alpha*} = -\frac{1}{|h_1|^2}\omega_{1\alpha}h_1 - \frac{1}{|h_2|^2}\omega_{2\alpha}h_2 + \Sigma_\beta \omega_{\alpha\beta}h_\beta, \quad \text{where} \quad (16)$$

$$\omega_{11} = -\frac{\lambda}{\theta r^2}d(\lambda r^2), \quad \omega_{12} = \frac{\lambda r^2}{\theta}(J \text{grad}(\lambda/r^2))^*, \quad \omega_{21} = -\frac{1}{\lambda r^2 \theta}(J \text{grad} \lambda)^*,$$

$$\omega_{22} = -\frac{1}{\lambda r^2 \theta}d\lambda, \quad \omega_{1\alpha} = r^2(A_\alpha \omega_1 - B_\alpha \omega_2), \quad \omega_{2\alpha} = -\frac{1}{\lambda}(C_\alpha \omega_1 + D_\alpha \omega_2),$$

$$\omega_{\alpha\beta} = \langle \nabla^\perp h_\alpha, h_\beta \rangle$$

where $\omega_i = X_i^*$, $i = 1, 2$, Z^* denotes the 1-form dual to $Z \in TM$. Also,

$$C_\alpha = \langle \nabla_{X_1}^\perp \delta, h_\alpha \rangle, \quad D_\alpha = \langle \nabla_{X_2}^\perp \delta, h_\alpha \rangle, \quad A_\alpha = \lambda_1 \psi_{12}^\alpha - \lambda_2 \psi_{11}^\alpha \quad \text{and} \quad B_\alpha = \lambda_1 \psi_{11}^\alpha + \lambda_2 \psi_{12}^\alpha.$$

Proof: A straightforward computation of the derivatives in the ambient space yields

$$\begin{aligned} \bar{\nabla}_{X_1}(\lambda_2\xi_1 - \lambda_1\xi_2) &= \mu f_*(-\lambda_2 X_1 + \lambda_1 X_2) - (\mu a_2 + \psi_{12}\lambda)\xi_1 + (\mu a_1 + \psi_{11}\lambda)\xi_2 \\ &\quad + X_2(\lambda)\delta + \Sigma_\alpha A_\alpha h_\alpha, \end{aligned}$$

$$\begin{aligned} \bar{\nabla}_{X_2}(\lambda_2\xi_1 - \lambda_1\xi_2) &= \mu f_*(\lambda_1 X_1 + \lambda_2 X_2) - (\mu a_1 - \psi_{11}\lambda)\xi_1 - (\mu a_2 - \psi_{12}\lambda)\xi_2 \\ &\quad - X_1(\lambda)\delta - \Sigma_\alpha B_\alpha h_\alpha. \end{aligned}$$

Another straightforward computation using (8), (9) and that

$$X_1(1/r^2) = -2\mu(a_1\lambda_1 + a_2\lambda_2), \quad X_2(1/r^2) = 2\mu(a_2\lambda_1 - a_1\lambda_2) \quad (17)$$

gives

$$\begin{aligned} h_{1*}X_1 &= g_*X_2 - \frac{\lambda}{\theta}X_1(\lambda r^2)h_1 - \frac{\lambda r^2}{\theta}X_2(\lambda/r^2)h_2 + r^2\Sigma_\alpha A_\alpha h_\alpha, \\ h_{1*}X_2 &= -g_*X_1 - \frac{\lambda}{\theta}X_2(\lambda r^2)h_1 + \frac{\lambda r^2}{\theta}X_1(\lambda/r^2)h_2 - r^2\Sigma_\alpha B_\alpha h_\alpha. \end{aligned}$$

Similarly, we have

$$\begin{aligned} h_{2*}X_1 &= g_*X_1 + \frac{1}{\lambda r^2\theta}(X_2(\lambda)h_1 - X_1(\lambda)h_2) - \frac{1}{\lambda}\Sigma_\alpha C_\alpha h_\alpha, \\ h_{2*}X_2 &= g_*X_2 - \frac{1}{\lambda r^2\theta}(X_1(\lambda)h_1 + X_2(\lambda)h_2) - \frac{1}{\lambda}\Sigma_\alpha D_\alpha h_\alpha, \end{aligned}$$

and (14) and (15) follow. The third equation is just the Weingarten formula. ■

We decompose h_1 and h_2 into its tangent and normal components to g , namely,

$$h_1 = g_*Y + \eta, \quad h_2 = g_*Z + \xi. \quad (18)$$

Lemma 9. *It holds that*

$$Y = J\text{grad}_g\varrho \quad \text{and} \quad Z = -\text{grad}_g\varrho, \quad (19)$$

where $\varrho = r^2/2$ and J denotes a complex structure in TM .

Proof: Let u be the conformal factor between the metrics induced by g and f on M^2 , that is, $\langle \cdot, \cdot \rangle_g = u\langle \cdot, \cdot \rangle_f$. From (8), we have

$$\nabla_{X_1}^\perp H = -\mu(a_1\xi_1 + a_2\xi_2) \quad \text{and} \quad \nabla_{X_2}^\perp H = \mu(a_2\xi_1 - a_1\xi_2). \quad (20)$$

We obtain using (9), (17) and (20) that

$$\begin{aligned} g_*Y &= \frac{1}{u}g_*(\langle h_1, g_*X_1 \rangle X_1 + \langle h_1, g_*X_2 \rangle X_2) \\ &= \frac{\mu r^4}{u}g_*((a_2\lambda_1 - a_1\lambda_2)X_1 + (a_1\lambda_1 + a_2\lambda_2)X_2) \\ &= \frac{r^4}{2u}g_*(X_2(1/r^2)X_1 - X_1(1/r^2)X_2) \\ &= -\frac{r^4}{2u}g_*J\text{grad}_f(1/r^2) \\ &= \frac{1}{u}g_*J\text{grad}_f\varrho. \end{aligned}$$

The computation of g_*Z is similar. ■

Lemma 10. *The vector fields η and ξ are linearly independent everywhere.*

Proof: Assume that $c_1\xi + c_2\eta = 0$ with $(c_1, c_2) \neq 0$ at $x \in M^2$. Then,

$$c_1h_1 + c_2h_2 = g_*(c_1Y + c_2Z).$$

Thus $g_*(c_1Z_1 + c_2Z_2)$ is normal to f at x and (9) implies that

$$c_1Z_1 + c_2Z_2 \in \ker(I - r^2A_H).$$

Since $\det(I - r^2A_H) = -r^4\theta^2\mu^2$, we conclude that $c_1Z_1 + c_2Z_2 = 0$. Hence $c_1h_1 + c_2h_2 = 0$, and this is a contradiction. ■

We now consider the orthogonal decomposition

$$N_gM = P \oplus \Lambda,$$

where $P = \text{span}\{\eta, \xi\}$ and $\Lambda = (N_1^f)^\perp \oplus L$ with $\dim L = 1$.

We observe that Y (and hence Z) cannot vanish. Otherwise, from Lemma 9 it follows that $|H|$ is constant. From (17) we obtain $a_1 = a_2 = 0$. Since f is S-Willmore, working as in the proof of Theorem 2, we see that the Ricci equation implies that (12) and (13) still hold. These immediately yield $\lambda_1 = \lambda_2 = 0$, which is a contradiction.

Lemma 11. *The surface f can be parametrized in terms of g as*

$$f = g - g_*\text{grad}_g\varrho - \rho\xi + \Omega w,$$

where $\rho = |\text{grad}_g\varrho|^2/|\xi|^2$, $\Omega = \sqrt{2\varrho - \rho(\rho + 1)|\xi|^2}$ and $w = -(H)^\Lambda/|(H)^\Lambda| \in L$.

Proof: Using (9) we have

$$\begin{aligned} r^2(H)^{g_*(TM)} &= \frac{r^2}{u}g_*(\langle H, g_*X_1 \rangle X_1 + \langle H, g_*X_2 \rangle X_2) \\ &= -\frac{r^4}{2u}g_*\text{grad}_f(1/r^2) \\ &= g_*\text{grad}_g\varrho. \end{aligned}$$

From (9), (18) and (19), we obtain that

$$\langle H, \eta \rangle = -\langle H, g_*Y \rangle = 0 \quad \text{and} \quad \langle H, \xi \rangle = -\langle H, g_*Z \rangle = \frac{1}{r^2}|\text{grad}_g\varrho|^2.$$

We also have $\langle \eta, \xi \rangle = 0$ from Lemma 9, and the result follows. ■

Lemma 12. *The mean curvature of g satisfies*

$$(H_g)^{\text{span}\{\eta\}} = 0 \quad \text{and} \quad (H_g)^{\text{span}\{\xi\}} = -2|h_2|^{-2}\xi.$$

Proof: Using Lemma 8, we have

$$d\omega_{11} = -\lambda\theta^{-2}d(1/r^2) \wedge d\lambda = \omega_{12} \wedge \omega_{21}.$$

Computing $d^2h_1 = 0$ using (14) to (16) gives

$$\begin{aligned} 0 &= d(g_* \circ J) + (g_* \circ J) \wedge \omega_{11} + g_* \wedge \omega_{12} \\ &\quad + (d\omega_{12} - \omega_{11} \wedge \omega_{12} - \omega_{12} \wedge \omega_{22} + |h_2|^{-2}\Sigma_\alpha \omega_{1\alpha} \wedge \omega_{2\alpha})h_2 \\ &\quad + \Sigma_\alpha (d\omega_{1\alpha} - \omega_{11} \wedge \omega_{1\alpha} - \omega_{12} \wedge \omega_{2\alpha} - \Sigma_\beta \omega_{1\alpha} \wedge \omega_{\alpha\beta})h_\alpha. \end{aligned}$$

From

$$\omega_{11}(X_1)X_1 + \omega_{11}(X_2)X_2 = -\frac{\lambda}{\theta r^2} \text{grad}(\lambda r^2), \quad \omega_{12}(X_2)X_1 - \omega_{12}(X_1)X_2 = \frac{\lambda r^2}{\theta} \text{grad}(\lambda/r^2)$$

we obtain that

$$(g_* \circ J) \wedge \omega_{11} + g_* \wedge \omega_{12} = \frac{4\lambda^2}{\theta r^2} g_* Z * 1,$$

where $*1$ is the volume element. Moreover, we have $d(g_* \circ J) = -2H_g * 1$. We obtain

$$\begin{aligned} 2H_g * 1 &= \frac{4\lambda^2}{\theta r^2} * 1 g_* Z + (d\omega_{12} - \omega_{11} \wedge \omega_{12} - \omega_{12} \wedge \omega_{22} + |h_2|^{-2}\Sigma_\alpha \omega_{1\alpha} \wedge \omega_{2\alpha})h_2 \\ &\quad + \Sigma_\alpha (d\omega_{1\alpha} - \omega_{11} \wedge \omega_{1\alpha} - \omega_{12} \wedge \omega_{2\alpha} - \Sigma_\beta \omega_{1\alpha} \wedge \omega_{\alpha\beta})h_\alpha. \end{aligned}$$

Hence H_g is perpendicular to η and the ξ -component of H_g is $-(2\lambda^2/\theta r^2)\xi$. ■

Proposition 13. *If a superconformal S -Willmore surface $f: M^2 \rightarrow \mathbb{R}^{n+2}$ is locally conformally substantial and free of minimal points, then the locus of centers g is a minimal surface if and only $n = 2$.*

Proof: If $n \geq 3$, it follows from Proposition 5 and Proposition 6 that $\dim N_1^f = 3$ and that $\theta \neq 0$ on an open dense subset of M^2 . Now, that g is not minimal is a consequence of Lemma 12. The case $n = 2$ follows from the result in [16], i.e., our Theorem 15. ■

We now prove the main result of this paper.

Proof of Theorem 1: In view of Corollary 4 we may denote $n_\alpha = h_\alpha \in N_g M$ for $\alpha \geq 3$. Differentiating (18) and using Lemma 8, gives

$$\nabla_X Y - A_\eta X = JX + \omega_{11}(X)Y + \omega_{12}(X)Z, \quad (21)$$

$$\nabla_X Z - A_\xi X = X + \omega_{21}(X)Y + \omega_{22}(X)Z, \quad (22)$$

$$A_{n_\alpha} X = \omega_{1\alpha}(X) \frac{Y}{|h_1|^2} + \omega_{2\alpha}(X) \frac{Z}{|h_2|^2}, \quad (23)$$

$$\alpha_g(X, Y) + \hat{\nabla}_X^\perp \eta = \omega_{11}(X)\eta + \omega_{12}(X)\xi + \Sigma_\alpha \omega_{1\alpha}(X)n_\alpha, \quad (24)$$

$$\alpha_g(X, Z) + \hat{\nabla}_X^\perp \xi = \omega_{21}(X)\eta + \omega_{22}(X)\xi + \Sigma_\alpha \omega_{2\alpha}(X)n_\alpha, \quad (25)$$

$$\hat{\nabla}_X^\perp n_\alpha = -\omega_{1\alpha}(X) \frac{\eta}{|h_1|^2} - \omega_{2\alpha}(X) \frac{\xi}{|h_2|^2} + \Sigma_\beta \omega_{\alpha\beta}(X)n_\beta, \quad (26)$$

where $\hat{\nabla}^\perp$ denotes the induced connection in the normal bundle of g .

We claim that

$$\begin{aligned} \langle A_\xi X, Z \rangle - X(\rho)|\xi|^2 - \frac{\rho}{2}X|\xi|^2 &= \langle A_\eta JX, Z \rangle - \rho \langle \hat{\nabla}_{JX}^\perp \xi, \eta \rangle \\ &= -\rho \langle A_\xi X, Z \rangle - \langle X, Z \rangle - \frac{1}{2}X|Z|^2. \end{aligned} \quad (27)$$

From the definition of the forms ω_{ij} we obtain

$$\omega_{11}(X) + \omega_{12}(JX) = \frac{4\langle Z, X \rangle}{|Z|^2 + |\xi|^2} \quad \text{and} \quad \omega_{21}(JX) = \omega_{22}(X). \quad (28)$$

Using (28) it follows from (21) and (22) that

$$\begin{aligned} \langle A_\xi Z, Z \rangle &= \langle \nabla_Z Z, Z \rangle - (\omega_{22}(Z) + 1)|Z|^2, \quad \langle A_\xi Y, Z \rangle = \langle \nabla_Y Z, Z \rangle - \omega_{22}(Y)|Z|^2, \\ \langle A_\xi Y, Y \rangle &= \langle \nabla_Y Z, Y \rangle + (\omega_{22}(Z) - 1)|Z|^2, \quad \langle A_\eta Z, Z \rangle = -\langle \nabla_Z Z, Y \rangle + \omega_{11}(Y)|Z|^2, \\ \langle A_\eta Y, Z \rangle &= -\langle \nabla_Y Z, Y \rangle - (1 + \omega_{11}(Z))|Z|^2 + \frac{4|Z|^4}{|Z|^2 + |\xi|^2}, \end{aligned}$$

whereas from (24) and (25) that

$$\begin{aligned} \langle A_\xi Z, Z \rangle &= -\langle \hat{\nabla}_Z^\perp \xi, \xi \rangle + \omega_{22}(Z)|\xi|^2, \\ \langle A_\xi Y, Z \rangle &= -\langle \hat{\nabla}_Y^\perp \xi, \xi \rangle + \omega_{22}(Y)|\xi|^2 = -\langle \hat{\nabla}_Z^\perp \eta, \xi \rangle - \omega_{11}(Y)|\xi|^2 \end{aligned}$$

and

$$\langle A_\xi Y, Y \rangle = -\langle \nabla_Y^\perp \eta, \xi \rangle + \omega_{11}(Z)|\xi|^2 - \frac{4|Z|^2|\xi|^2}{|Z|^2 + |\xi|^2}.$$

Then,

$$\rho \langle \hat{\nabla}_Y^\perp \xi, \xi \rangle = -(\rho + 1) \langle A_\xi Y, Z \rangle + \langle \nabla_Y Z, Z \rangle, \quad (29)$$

$$\rho \langle \hat{\nabla}_Z^\perp \xi, \xi \rangle = -(\rho + 1) \langle A_\xi Z, Z \rangle + \langle \nabla_Z Z, Z \rangle - |Z|^2, \quad (30)$$

$$\rho \langle \hat{\nabla}_Z^\perp \xi, \eta \rangle = \rho \langle A_\xi Y, Z \rangle + \langle A_\eta Z, Z \rangle - \langle \nabla_Z Y, Z \rangle, \quad (31)$$

$$\rho\langle\hat{\nabla}_Y^\perp\xi,\eta\rangle=\rho\langle A_\xi Y,Y\rangle+\langle A_\eta Y,Z\rangle-\langle\nabla_Y Y,Z\rangle+|Z|^2. \quad (32)$$

and

$$(2+\mathrm{tr}A_\xi)|Z|^2=\langle\nabla_Z Z,Z\rangle-\langle\nabla_Y Y,Z\rangle. \quad (33)$$

It is enough to argue for $X=Y$ and $X=Z$. We have using (19) and (30) that

$$\langle A_\xi Y,Z\rangle-Y(\rho)|\xi|^2-\rho\langle\hat{\nabla}_Y^\perp\xi,\xi\rangle=-\rho\langle A_\xi Y,Z\rangle-\langle\nabla_Y Z,Z\rangle. \quad (34)$$

Similarly, from (19) and (30) we also obtain

$$\langle A_\xi Z,Z\rangle-Z(\rho)|\xi|^2-\rho\langle\hat{\nabla}_Z^\perp\xi,\xi\rangle=-\rho\langle A_\xi Z,Z\rangle-\langle\nabla_Z Z,Z\rangle-|Z|^2, \quad (35)$$

and one equality in (27) follows from (34) and (35). Using (19) and (31), we have

$$\langle A_\eta JY,Z\rangle-\rho\langle\hat{\nabla}_{JY}^\perp\xi,\eta\rangle=-\rho\langle A_\xi Y,Z\rangle-\langle\nabla_Y Z,Z\rangle.$$

Moreover, using (33) and (32) we obtain

$$\begin{aligned} \langle A_\eta JZ,Z\rangle-\rho\langle\hat{\nabla}_{JZ}^\perp\xi,\eta\rangle &= |Z|^2+\rho\langle A_\xi Y,Y\rangle-\langle\nabla_Y Y,Z\rangle \\ &= -\rho\langle A_\xi Z,Z\rangle+|Z|^2+\rho|Z|^2\mathrm{tr}A_\xi-\langle\nabla_Y Y,Z\rangle \\ &= -\rho\langle A_\xi Z,Z\rangle-\langle\nabla_Z Z,Z\rangle+|Z|^2((\rho+1)\mathrm{tr}A_\xi+3). \end{aligned}$$

Since Lemma 12 gives $\mathrm{tr}A_\xi=-4/(\rho+1)$, we obtain

$$\langle A_\eta JZ,Z\rangle-\rho\langle\hat{\nabla}_{JZ}^\perp\xi,\eta\rangle=-\rho\langle A_\xi Z,Z\rangle-\langle\nabla_Z Z,Z\rangle-|Z|^2,$$

and this completes the proof of (27).

Assume that f is as in the statement. From Lemma 11 we have $f=g+g_*Z-\rho\xi+\Omega w$. Then, define $f^-:M^2\rightarrow\mathbb{R}^{n+2}$ by

$$f^-=g+g_*Z-\rho\xi+\Omega w_-$$

where $w_-=-w$. We show that

$$N_{f^-}M=\mathrm{span}\{h_1,h_2,h,h_3,\dots,h_{n-1}\}$$

where $h=g_*\nabla r+(\rho/r)\xi-(\Omega/r)w_-$. First compute f_*^- and use (21) to (26) to obtain

$$\langle f_*^-X,h_1\rangle=\langle f_*^-X,h_2\rangle=\langle f_*^-X,h_\alpha\rangle=0.$$

To prove that also h is normal to f^- it is sufficient to see that h is unitary and that $f^-=g-rh$. Observe also that (16) implies that $N_1^{f^-}=\mathrm{span}\{h_1,h_2,h\}$ and that the shape operators of f^- satisfy $A_{h_\alpha}^-=0$.

To prove that f^- is superconformal we need to show that there exist an orthogonal tangent basis $X_1, X_2 = JX_1$ and functions a, b such that

$$(h_{1*}X_1)^{f_*^-(TM)} = af_*^-X_1 + bf_*^-X_2, \quad (36)$$

$$(h_{1*}X_2)^{f_*^-(TM)} = bf_*^-X_1 - af_*^-X_2, \quad (37)$$

$$(h_{2*}X_1)^{f_*^-(TM)} = -bf_*^-X_1 + af_*^-X_2, \quad (38)$$

$$(h_{2*}X_2)^{f_*^-(TM)} = af_*^-X_1 + bf_*^-X_2, \quad (39)$$

$$(h_*X_1)^{f_*^-(TM)} = \frac{1}{r}(-(1+b)f_*^-X_1 + af_*^-X_2), \quad (40)$$

$$(h_*X_2)^{f_*^-(TM)} = \frac{1}{r}(af_*^-X_1 - (1-b)f_*^-X_2). \quad (41)$$

Using (14)-(16), we see that

$$\begin{aligned} (h_{1*}X)^{f_*^-(TM)} &= \frac{|\eta|^2}{|h_1|^2}g_*JX - \frac{\langle JX, Y \rangle}{|h_1|^2}\eta - \frac{|\eta|^2\langle JX, Z \rangle}{|\xi|^2|h_1|^2}\xi - \frac{\langle JX, Z \rangle}{r^2}\Omega w_-, \\ (h_{2*}X)^{f_*^-(TM)} &= \frac{|\eta|^2}{|h_1|^2}g_*X - \frac{\langle X, Y \rangle}{|h_1|^2}\eta - \frac{|\eta|^2\langle X, Z \rangle}{|\xi|^2|h_1|^2}\xi - \frac{\langle X, Z \rangle}{r^2}\Omega w_- \end{aligned} \quad (42)$$

and

$$(h_*X)^{f_*^-(TM)} = \frac{1}{r}(-f_*^-X + (h_{2*}X)^{f_*^-(TM)}).$$

Thus

$$(h_{1*}X_1)^{f_*^-(TM)} = (h_{2*}X_2)^{f_*^-(TM)} \quad \text{and} \quad (h_{1*}X_2)^{f_*^-(TM)} + (h_{2*}X_1)^{f_*^-(TM)} = 0.$$

This means that (36)-(41) are equivalent to (38) and (39), and we only have to choose the basis so that (38) and (39) hold or, equivalently, that

$$f_*^-X_1 = c(h_{2*}X_1)^{f_*^-(TM)} + d(h_{2*}X_2)^{f_*^-(TM)}, \quad f_*^-X_2 = d(h_{2*}X_1)^{f_*^-(TM)} - c(h_{2*}X_2)^{f_*^-(TM)}. \quad (43)$$

From Lemma 12 it follows that the self adjoint tensor field L given by

$$LX = X + \nabla_X Z + \rho A_\xi X - \Omega A_{w_-} X$$

has zero trace. Let $\{X_1, X_2 = JX_1\}$ be an orthonormal basis with respect to the metric induced by g . Clearly, for suitable functions c, d , we have

$$LX_1 = \frac{|\eta|^2}{|Z|^2 + |\eta|^2}(cX_1 + dX_2), \quad LX_2 = \frac{|\eta|^2}{|Z|^2 + |\eta|^2}(dX_1 - cX_2), \quad (44)$$

which are actually the $g_*(TM)$ -components of (43).

Using (27), (24) and (25) we see that the ξ and η components of (43) are equivalent to

$$\langle LX_1, Z \rangle = \frac{|\eta|^2}{|Z|^2 + |\eta|^2} \langle cX_1 + dX_2, Z \rangle, \quad \langle LX_2, Z \rangle = \frac{|\eta|^2}{|Z|^2 + |\eta|^2} \langle dX_1 - cX_2, Z \rangle,$$

which are part of (44).

The w -components of (43) are equivalent to

$$\begin{aligned} \langle A_{w-} X_1, Z \rangle + \frac{X_1(\Omega)}{1 + \rho} &= -\frac{\Omega}{r^2(1 + \rho)} \langle cX_1 + dX_2, Z \rangle, \\ \langle A_{w-} X_2, Z \rangle + \frac{X_2(\Omega)}{1 + \rho} &= -\frac{\Omega}{r^2(1 + \rho)} \langle dX_1 - cX_2, Z \rangle. \end{aligned}$$

On account of (44) and

$$\frac{|\eta|^2}{|Z|^2 + |\eta|^2} = \frac{\Omega^2}{r^2(1 + \rho)}$$

the above equations are equivalent to

$$\langle X + \nabla_X Z + \rho A_\xi X, Z \rangle + \frac{1}{1 + \rho} \Omega X(\Omega) = 0.$$

We now argue that this holds. Indeed, this follows by differentiating $|Z|^2(1 + \rho) + \Omega^2 = r^2$ with respect to X and using (22) and (25).

Finally we note that the $(N_1^f)^\perp$ -components validity of (43) follows from

$$N_{f-} M = \text{span}\{h_1, h_2, h, h_3, \dots, h_{n-1}\},$$

equation (42) and Corollary 4.

To complete the proof that f^- is superconformal, we need to show that the basis $\{X_1, X_2\}$ can be chosen to be orthonormal with respect to g . An easy computation gives

$$|(h_{2*} X_1)^{f_*^- (TM)}|^2 = |(h_{2*} X_2)^{f_*^- (TM)}|^2 = \frac{|\eta|^2}{|Z|^2 + |\eta|^2}$$

and

$$\langle (h_{2*} X_1)^{f_*^- (TM)}, (h_{2*} X_2)^{f_*^- (TM)} \rangle = 0.$$

Thus, in view of (43) we have to show that

$$|f_*^- X_1|^2 = |f_*^- X_2|^2 = \frac{|\eta|^2}{|Z|^2 + |\eta|^2} (c^2 + d^2) \quad \text{and} \quad \langle f_*^- X_1, f_*^- X_2 \rangle = 0.$$

Hence f^- and g are conformal and the desired orthonormal basis with respect to the metric induced by f^- is

$$Y_j = \frac{1}{|\eta|} (|Z|^2 + |\eta|^2)^{1/2} (c^2 + d^2)^{1/2} X_j, \quad j = 1, 2.$$

In particular, the mean curvature vector field of f^- is given by $H_- = (1/r)h$ and the locus of the centers of the corresponding central sphere congruence is

$$f^- + \frac{1}{|H_-|^2} H_- = f^- + rh = g.$$

To conclude the proof that f^- is the dual to f it remains to show that

$$f_*(TM) \oplus \text{span}\{H\} = f_*^-(TM) \oplus \text{span}\{H_-\}$$

which follows from

$$(f_*(TM) \oplus \text{span}\{H\})^\perp = \text{span}\{h_1, h_2, h_3, \dots, h_{n-1}\} = (f_*^-(TM) \oplus \text{span}\{H_-\})^\perp.$$

Conversely, if f allows a dual superconformal surface then it is S-Willmore by a result of Ma [12]. According to Lemma 11, we have $f = g + g_*Z - \rho\xi + \Omega w$ and the dual to f is given by

$$\tilde{f} = g + g_*Z - \rho\xi - \Omega w.$$

Then, the parametrization

$$\tilde{f} = f + \frac{2}{|H|^2} (H)^\Lambda$$

of the dual follows easily from $\Omega w = -r^2(H)^\Lambda$, $\Lambda = (N_1^f)^\perp \oplus L$ and

$$N_1^f = \text{span}\{h_1, h_2\} \oplus \text{span}\{H\}.$$

Assume the dual reduces to a point p_0 , i.e., $f = p_0 + 2\Omega w$. On the other hand, from

$$w^\perp = \langle w, h_1 \rangle \frac{h_1}{|h_1|^2} + \langle w, h_2 \rangle \frac{h_2}{|h_2|^2} + \langle w, H \rangle \frac{H}{|H|^2} + \Sigma_\alpha \langle w, \eta_\alpha \rangle \eta_\alpha$$

and Lemma 11 we obtain that

$$w^\perp = \langle w, r^2(H)^\Lambda \rangle H = -\Omega H.$$

Thus,

$$(f - p_0)^\perp = \varphi H, \quad \varphi = -2\Omega^2.$$

Moreover, we have

$$|f - p_0|^2 + 2\varphi = 0.$$

Consider the inversion \mathcal{I} with respect to a sphere with radius $R = 1$ centered at p_0 and the immersion $\tilde{f} = \mathcal{I} \circ f$. Then, there is a vector bundle isometry \mathcal{P} between the normal bundles $N_f M$ and $N_{\tilde{f}} M$ (see [6]) given by

$$\mathcal{P}\mu = \mu - 2 \frac{\langle f - p_0, \mu \rangle}{|f - p_0|^2} (f - p_0)$$

such that shape operators of f and \tilde{f} are related by

$$\tilde{A}_{\mathcal{P}\mu} = |f - p_0|^2 A_\mu + 2\langle f - p_0, \mu \rangle I.$$

We can easily find that the mean curvature vector of \tilde{f} is given by

$$H_{\tilde{f}} = \mathcal{P}(|f - p_0|^2 H + 2(f - p_0)^\perp).$$

Using $(f - p_0)^\perp = \varphi H$ and $|f - p_0|^2 + 2\varphi = 0$, we deduce that \tilde{f} is minimal in \mathbb{R}^{n+2} .

Conversely, assume that the surface is a composition of a 1-isotropic surface with an inversion with respect to the sphere in \mathbb{R}^{n+2} with radius $R = 1$ centered at p_0 . Then,

$$(f - p_0)^\perp = \varphi H \quad \text{and} \quad \varphi = -\frac{1}{2}\langle f - p_0, f - p_0 \rangle.$$

From this we obtain that

$$\alpha(X, \text{grad } \varphi) = X(\varphi)H + \varphi \nabla_X^\perp H.$$

Using (9) we see that $f - p_0$ is perpendicular to the surface g . Since it is also perpendicular to h_1, h_2 , it follows that it is perpendicular to the plane bundle P . Furthermore, it is perpendicular to $(N_1^f)^\perp$. Thus, we conclude that $f - p_0 = \sigma w$. From this we obtain

$$\frac{1}{\sigma}(f - p_0)^\perp = \langle w, h_1 \rangle \frac{h_1}{|h_1|^2} + \langle w, h_2 \rangle \frac{h_2}{|h_2|^2} + \langle w, H \rangle \frac{H}{|H|^2} + \Sigma_\alpha \langle w, \eta_\alpha \rangle \eta_\alpha$$

or, using Lemma 11, that

$$(f - p_0)^\perp = \sigma \langle w, (r^2 H)^\wedge \rangle H = -\sigma \Omega H.$$

Hence $\varphi = -\sigma \Omega$. In view of $\sigma^2 = -2\varphi$, we have that $\Omega = |f - p_0|/2$ and $\sigma = |f - p_0|$. Thus $f - p_0 = 2\Omega w$, which shows that the dual to f reduces to the point p_0 . Finally, the cases (ii) and (iii) follow directly from Proposition 14 given next. ■

In the following result we do not assume that the surface is superconformal.

Proposition 14. *Let $f: M^2 \rightarrow \mathbb{R}^{n+2}$ be a surface with dual surface $\tilde{f} = T \circ f$ where T denotes a conformal diffeomorphism of \mathbb{R}^{n+2} .*

- (i) *Then T is composition of an inversion with a reflection with respect to the center of the inversion if and only if f is either a composition of a minimal surface in the sphere $\mathbb{S}^{n+2} \subset \mathbb{R}^{n+3}$ with a stereographic projection onto \mathbb{R}^{n+2} or a minimal surface in a sphere $\mathbb{S}^{n+1} \subset \mathbb{R}^{n+2}$.*
- (ii) *Then T is an inversion if and only if f is a composition of a minimal surface in hyperbolic space $\mathbb{H}^{n+2} \subset \mathbb{L}^{n+3}$ with a stereographic projection onto \mathbb{R}^{n+2} .*

Proof: A surface $\tilde{f}: M^2 \rightarrow \mathbb{R}^{n+2}$ is the dual to a surface $f: M^2 \rightarrow \mathbb{R}^{n+2}$ if

$$f + r^2 H = \tilde{f} + r^2 \tilde{H}, \quad (45)$$

$$f_*(TM) \oplus \text{span}\{H\} = \tilde{f}_*(TM) \oplus \text{span}\{\tilde{H}\} \quad (46)$$

and $|H| = |\tilde{H}| = 1/r$.

To prove (i) first assume that $\tilde{f} = -\mathcal{I} \circ f$, where \mathcal{I} is the inversion with respect to a sphere with radius 1 centered at the origin. The mean curvature vector field \tilde{H} is given by $\tilde{H} = -\hat{H}$, where \hat{H} is the mean curvature of the surface $\hat{f} = \mathcal{I} \circ f = f/|f|^2$. From the results in [6] there is a vector bundle isometry \mathcal{P} between the normal bundles $N_f M$ and $N_{\hat{f}} M$ given by

$$\mathcal{P}\mu = \mu - \frac{2}{|f|^2} \langle f, \mu \rangle f$$

such that shape operators of f and \hat{f} are related by

$$\hat{A}_{\mathcal{P}\mu} = |f|^2 A_\mu + 2 \langle f, \mu \rangle I$$

and the mean curvature vectors by

$$\hat{H} = \mathcal{P}(|f|^2 H + 2f^\perp).$$

Using (45) we deduce that

$$(1 - 2r^2 \langle f, H \rangle + \frac{1}{|f|^2} (1 - 4r^2 |f^\perp|^2)) f = -r^2 ((1 + |f|^2) H + 2f^\perp).$$

Thus, the left hand side must vanish unless $f \in N_f M$. In the latter case f is a minimal surface in a sphere. If not we have

$$2f^\perp = -(1 + |f|^2) H.$$

Let e be a unit vector in $\mathbb{R}^{n+3} = \mathbb{R}^{n+2} \oplus \mathbb{R}e$ and let \mathcal{T} be the inversion

$$\mathcal{T}(p) = q_0 + \frac{1}{|p - q_0|^2} (p - q_0)$$

with respect to the sphere \mathbb{S}^{n+2} with radius 1 centered at $q_0 = (0, 1) = e$. If $\bar{f} = \mathcal{T} \circ f$, there is, as before, a vector bundle isometry $\bar{\mathcal{P}}$ between $N_f M$ and $N_{\bar{f}} M$ given by

$$\bar{\mathcal{P}}\mu = \mu - \frac{2}{|f - q_0|^2} \langle f - q_0, \mu \rangle (f - q_0)$$

such that shape operators of f and \bar{f} are related by

$$\bar{A}_{\bar{\mathcal{P}}\mu} = |f - q_0|^2 A_\mu + 2 \langle f - q_0, \mu \rangle I$$

and the mean curvature vector field of \bar{f} is given by

$$\begin{aligned}\bar{H} &= \bar{\mathcal{P}}(|f - q_0|^2 H + 2(f - q_0)^\perp) \\ &= \bar{\mathcal{P}}(|f - q_0|^2 H - (1 + |f|^2)H - 2e) \\ &= -2\bar{\mathcal{P}}e \\ &= -4(\bar{f} - (1/2)e).\end{aligned}$$

Thus \bar{f} is minimal in the sphere $\mathbb{S}_{1/2}^{n+2}(e/2)$ with radius $1/2$ centered at $e/2$.

Conversely, let f be a composition of a minimal surface in the sphere $\mathbb{S}_{1/2}^{n+2}(e/2)$ in $\mathbb{R}^{n+3} = \mathbb{R}^{n+2} \oplus \mathbb{R}e$ with a stereographic projection onto \mathbb{R}^{n+2} . Then,

$$f^\perp = -\frac{1}{2}|f - q_0|^2 H = -\frac{1}{2}(1 + |f|^2)H$$

where $q_0 = e$. Consider the surface $\tilde{f} = -\mathcal{I} \circ f$, where \mathcal{I} is the inversion with respect to a sphere with radius 1 centered at the origin. We claim that \tilde{f} is the dual to f . Consider the surface $\hat{f} = f/|f|^2$ with mean curvature \hat{H} and \mathcal{P} the corresponding bundle isometry between the normal bundles of f and \hat{f} . The mean curvature of \tilde{f} is given by

$$\tilde{H} = -\hat{H} = -\mathcal{P}(|f|^2 H + 2f^\perp) = \mathcal{P}(H).$$

We have that $|\tilde{H}| = |H|$ and

$$\begin{aligned}\tilde{f} + r^2 \tilde{H} &= -\frac{f}{|f|^2} + r^2 \mathcal{P}(H) \\ &= -\frac{f}{|f|^2} + r^2 H - \frac{2r^2}{|f|^2} \langle f^\perp, H \rangle f \\ &= -\frac{f}{|f|^2} + r^2 H + \frac{1 + |f|^2}{|f|^2} f \\ &= f + r^2 H.\end{aligned}$$

It remains to show that (46) holds or, equivalently, that

$$f_*(TM) \oplus \text{span}\{H\} = \mathcal{P}(f_*(TM) \oplus \text{span}\{H\}),$$

since

$$\tilde{f}_* = -\frac{1}{|f|^2} \mathcal{P} \circ f_*.$$

Now decompose f in its tangent and normal components as $f = f_* V + f^\perp$. Then,

$$\begin{aligned}\mathcal{P}H &= H - \frac{2}{|f|^2} \langle f^\perp, H \rangle f \\ &= -\frac{2}{|f|^2} \langle f, H \rangle f_* V + \left(1 + \frac{1}{|f|^2} \langle f, H \rangle (1 + |f|^2)\right) H \in f_*(TM) \oplus \text{span}\{H\}.\end{aligned}$$

Moreover,

$$\begin{aligned}
\mathcal{P}f_*X &= f_*X - \frac{2}{|f|^2}\langle f, f_*X \rangle f \\
&= f_*(X - 2\langle X, V \rangle V) - \frac{2}{|f|^2}\langle X, V \rangle f^\perp \\
&= f_*(X - 2\langle X, V \rangle V) + \frac{1}{|f|^2}\langle X, V \rangle (1 + |f|^2)H \in f_*(TM) \oplus \text{span}\{H\}.
\end{aligned}$$

Thus \tilde{f} is the dual to f .

To prove (ii), we proceed as in the previous case assuming that $\tilde{f} = \mathcal{I} \circ f$ with \mathcal{I} as before, and find that $f^\perp = \frac{1}{2}(1 - |f|^2)H$. Now let e be a vector in the Lorentzian space $\mathbb{L}^{n+3} = \mathbb{R}^{n+2} \oplus \mathbb{R}e$ such that $\langle e, e \rangle = -1$. Then, the “inversion” \mathcal{T} with respect to the hyperbolic space

$$\mathbb{H}_1^{n+2}(q_0) = \{p \in \mathbb{L}^{n+3} : \langle p - q_0, p - q_0 \rangle = -1\}$$

is given by

$$\mathcal{T}(p) = q_0 - \frac{1}{\langle p - q_0, p - q_0 \rangle}(p - q_0)$$

with $q_0 = (0, 1) = e$. If $\hat{f} = \mathcal{T} \circ f$, there is a vector bundle isometry $\hat{\mathcal{P}}$ between the normal bundles N_fM and $N_{\hat{f}}M$ given by

$$\hat{\mathcal{P}}\mu = \mu - 2\frac{\langle f - q_0, \mu \rangle}{\langle f - q_0, f - q_0 \rangle}(f - q_0)$$

such that shape operators of f and \tilde{f} are related by

$$\hat{A}_{\hat{\mathcal{P}}\mu} = -\langle f - q_0, f - q_0 \rangle A_\mu - 2\langle f - q_0, \mu \rangle I$$

and the mean curvature vector field of $\hat{f} = \mathcal{T} \circ f$ is given by

$$\begin{aligned}
H_{\hat{f}} &= -\hat{\mathcal{P}}(\langle f - q_0, f - q_0 \rangle H + 2(f - q_0)^\perp) \\
&= 2\hat{\mathcal{P}}e \\
&= 4(\hat{f} - e/2),
\end{aligned}$$

and thus \hat{f} is minimal in

$$\mathbb{H}_{1/2}^{n+2}(e/2) = \mathbb{H}_R^{n+2}(q_0) = \{p \in \mathbb{L}^{n+3} : \langle p - q_0, p - q_0 \rangle = -1/4\}.$$

Conversely, assume that f is a composition of a minimal surface in a $\mathbb{H}_{1/2}^{n+2}(e/2)$ in the Lorentzian space $\mathbb{L}^{n+3} = \mathbb{R}^{n+2} \oplus \mathbb{R}e$ with a stereographic projection onto \mathbb{R}^{n+2} . Then, we have

$$f^\perp = \frac{1}{2}(1 - |f|^2)H.$$

Consider the surface $\tilde{f} = \mathcal{I} \circ f$, where \mathcal{I} is the inversion with respect to a sphere with radius $R = 1$ centered at the origin. We claim that \tilde{f} is the dual to f . Its mean curvature is given by

$$\tilde{H} = \mathcal{P}(|f|^2 H + 2f^\perp) = \mathcal{P}(H),$$

where \mathcal{P} is the corresponding bundle isometry from $N_f M$ to $N_{\tilde{f}} M$. Then $|\tilde{H}| = |H|$ and

$$\begin{aligned} \tilde{f} + r^2 \tilde{H} &= \frac{f}{|f|^2} + r^2 \mathcal{P}H \\ &= \frac{f}{|f|^2} + r^2 H - \frac{2r^2 \langle f^\perp, H \rangle}{|f|^2} f \\ &= \frac{f}{|f|^2} + r^2 H - \frac{1 - |f|^2}{|f|^2} f \\ &= f + r^2 H. \end{aligned}$$

The proof that (46) holds is the same as before, and thus \tilde{f} is the dual to f . ■

3 The codimension two case

In this section, we give a proof of the main result for codimension two in [7] making use of the computations of this paper.

For a surface $g: M^2 \rightarrow \mathbb{R}^4$ consider the two possible complex structures \hat{J}_\pm on $N_g M$ and denote by \mathcal{J}_\pm the complex structure on the induced bundle $g^*(T\mathbb{R}^4)$ given by

$$\mathcal{J}_\pm \circ g_* = g_* \circ J \quad \text{and} \quad \mathcal{J}_\pm|_{N_g M} = \hat{J}_\pm.$$

Theorem 15. *Let $f: M^2 \rightarrow \mathbb{R}^4$ be a superconformal surface free of minimal and umbilical points. Then its surface of centers g is minimal and*

$$f = g + \mathcal{J}_\pm h, \tag{47}$$

where h is the conjugate minimal surface to g . Conversely, given a simply connected minimal surface g with conjugate surface h , then (47) parametrizes a superconformal surface.

Proof: First assume that $f: M^2 \rightarrow \mathbb{R}^4$ is a superconformal surface and define

$$h = r^2(\lambda_2 \xi_1 - \lambda_1 \xi_2).$$

Arguing as in Lemma 8, we obtain

$$g_* X_1 = -\frac{\mu}{H} f_*(X_1) + \frac{\mu}{H^2} (a_1 \xi_1 - a_2 \xi_2), \quad g_* X_2 = \frac{\mu}{H} f_*(X_2) - \frac{\mu}{H^2} (a_2 \xi_1 + a_1 \xi_2).$$

Moreover,

$$h_*X_1 = g_*X_2 \quad \text{and} \quad h_*X_2 = -g_*X_1.$$

This shows that $h_* = g_* \circ J$. In particular, g is minimal and h is the conjugate surface. We decompose h into its tangent and normal components

$$h = g_*(Y) + \eta. \tag{48}$$

From Lemma 9 we know that $Y = J \text{grad}_g \varrho$. Using (9), we easily obtain

$$\langle r^2H + g_*(JY), g_*X \rangle = 0$$

for any $X \in TM$. Since η is perpendicular to H , we have

$$\langle r^2H + g_*(JY), \eta \rangle = 0.$$

Hence, $r^2H + g_*(JY) \in N_gM$ is perpendicular to η . Moreover, it is easy to see that

$$|r^2H + g_*(JY)| = |\eta|.$$

This means that

$$r^2H + g_*(JY) = -\hat{J}_\pm \eta.$$

Using (9), (17) and (20), we see that the tangent component of r^2H is given by

$$\begin{aligned} (r^2H)^{g_*(TM)} &= \frac{r^2}{u} g_*(\langle H, g_*X_1 \rangle X_1 + \langle H, g_*X_2 \rangle X_2) \\ &= \frac{\mu r^4}{u} g_*(\lambda_1 a_1 X_1 - \lambda_1 a_2 X_2) \\ &= -\frac{\mu r^4}{u} g_*\left(\frac{X_1(H^2)}{2\mu} X_1 + \frac{X_2(H^2)}{2\mu} X_2\right) \\ &= -\frac{r^4}{2u} g_*(\text{grad } H^2) = \frac{1}{2u} g_*(\text{grad } r^2) \\ &= \frac{1}{2} g_*(\text{grad}_g r^2) \\ &= -g_*(JY). \end{aligned}$$

Hence, we obtain

$$f = g + g_*(JY) + \hat{J}_\pm \eta.$$

For the converse, let g be a simply connected minimal surface with conjugate h . We claim that $f = g + \mathcal{J}_\pm h$ is superconformal. We decompose h as in (48). Differentiating with respect to $X \in TM$ and using that $h_* = g_* \circ J$ yields

$$JX = \nabla_X Y - A_\eta X, \quad \alpha_g(X, Y) = -\nabla_X^\perp \eta.$$

Then, from $f = g + g_*(JY) + \hat{J}_\pm \eta$ we find

$$f_*X = g_*(J \circ A_\eta X - A_{\hat{J}_\pm \eta} X) + \alpha_g(X, JY) - \hat{J}_\pm \alpha_g(X, Y).$$

Moreover, for any $X_1, X_2 \in TM$, we have

$$\begin{aligned} \langle f_*X_1, f_*X_2 \rangle &= -(\det A_\eta + \det A_{\hat{J}_\pm \eta}) \langle X_1, X_2 \rangle_g - \langle JA_\eta X_1, A_{\hat{J}_\pm \eta} X_2 \rangle - \langle A_{\hat{J}_\pm \eta} X_1, JA_\eta X_2 \rangle \\ &\quad + \langle \alpha_g(X_1, Y), \alpha_g(X_2, Y) \rangle + \langle \alpha_g(X_1, JY), \alpha_g(X_2, JY) \rangle \\ &\quad - \langle \alpha_g(X_1, JY), \hat{J}_\pm \alpha_g(X_2, Y) \rangle - \langle \alpha_g(X_2, JY), \hat{J}_\pm \alpha_g(X_1, Y) \rangle. \end{aligned}$$

The Gaussian curvature K and the normal curvature K^\perp of g satisfy

$$\langle JA_\eta X_1, A_{\hat{J}_\pm \eta} X_2 \rangle + \langle JA_\eta X_2, A_{\hat{J}_\pm \eta} X_1 \rangle = |\eta|^2 K^\perp \langle X_1, X_2 \rangle_g,$$

$$\langle \alpha_g(X_1, JY), \hat{J}_\pm \alpha_g(X_2, Y) \rangle + \langle \alpha_g(X_2, JY), \hat{J}_\pm \alpha_g(X_1, Y) \rangle = |Y|^2 K^\perp \langle X_1, X_2 \rangle_g,$$

and

$$\langle \alpha_g(X_1, Y), \alpha_g(X_2, Y) \rangle + \langle \alpha_g(X_1, JY), \alpha_g(X_2, JY) \rangle = -|Y|^2 K \langle X_1, X_2 \rangle_g.$$

Thus, the induced metric of f is given by

$$\langle \cdot, \cdot \rangle_f = -|h|^2 (K + K^\perp) \langle \cdot, \cdot \rangle_g.$$

The normal bundle of f is spanned by h and $\mathcal{J}_\pm h$. Moreover,

$$\begin{aligned} \tilde{\nabla}_X f_* V &= g_*(\nabla_X JA_\eta V - \nabla_X A_{\hat{J}_\pm \eta} V - A_{\alpha_g(JY, V)} X + A_{\hat{J}_\pm \alpha_g(Y, V)} X) \\ &\quad + \alpha_g(JX, A_\eta V) - \alpha_g(X, A_{\hat{J}_\pm \eta} V) + \nabla_X^\perp \alpha_g(JY, V) - \hat{J}_\pm \nabla_X^\perp \alpha_g(Y, V). \end{aligned}$$

Since g is minimal, we have $J \circ A_\nu = -A_\nu \circ J$ for any $\nu \in N_g M$ and $A_\nu \partial_z \in T''M$ for any local complex coordinate z . Since the (1,1)-part of the second fundamental form of g vanishes, we obtain

$$\alpha_g(\partial_z, A_\nu \partial_z) = 0.$$

We now easily see that

$$\begin{aligned} \tilde{\nabla}_{\partial_z} f_* \partial_z &= g_*(-i \nabla_{\partial_z} A_\eta \partial_z - \nabla_{\partial_z} A_{\hat{J}_\pm \eta} \partial_z - i A_{\alpha_g(Y, \partial_z)} \partial_z + A_{\hat{J}_\pm \alpha_g(Y, \partial_z)} \partial_z) \\ &\quad + i \nabla_{\partial_z}^\perp \alpha_g(Y, \partial_z) - \hat{J}_\pm \nabla_{\partial_z}^\perp \alpha_g(Y, \partial_z). \end{aligned}$$

Thus, the second fundamental form of f satisfies

$$\begin{aligned} \langle \alpha_f(\partial_z, \partial_z), h \rangle &= \langle \tilde{\nabla}_{\partial_z} f_* \partial_z, h \rangle \\ &= -i \langle \nabla_{\partial_z} A_\eta \partial_z, Y \rangle - \langle \nabla_{\partial_z} A_{\hat{J}_\pm \eta} \partial_z, Y \rangle - i \langle A_{\alpha_g(\partial_z, Y)} \partial_z, Y \rangle \\ &\quad + \langle A_{\hat{J}_\pm \alpha_g(\partial_z, Y)} \partial_z, Y \rangle + i \langle \nabla_{\partial_z}^\perp \alpha_g(\partial_z, Y), \eta \rangle - \langle \hat{J}_\pm \nabla_{\partial_z}^\perp \alpha_g(\partial_z, Y), \eta \rangle \end{aligned}$$

and

$$\begin{aligned}\langle \alpha_f(\partial_z, \partial_z), \mathcal{J}_\pm h \rangle &= \langle \nabla_{\partial_z} A_\eta \partial_z, Y \rangle - i \langle \nabla_{\partial_z} A_{\hat{J}_\pm \eta} \partial_z, Y \rangle + \langle a_{\alpha_g(\partial_z, Y)} \partial_z, Y \rangle \\ &\quad + i \langle a_{\hat{J}_\pm \alpha_g(\partial_z, Y)} \partial_z, Y \rangle + \langle \nabla_{\partial_z}^\perp \alpha_g(\partial_z, Y), \hat{J}_\pm \eta \rangle - \langle \nabla_{\partial_z}^\perp \alpha_g(\partial_z, Y), \eta \rangle.\end{aligned}$$

Hence,

$$\langle \alpha_f(\partial_z, \partial_z), \mathcal{J}_\pm h \rangle = i \langle \alpha_f(\partial_z, \partial_z), h \rangle.$$

This means that $\alpha_f(\partial_z, \partial_z)$ is isotropic, and thus f is superconformal. ■

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